

K3 Orientifolds

Eric G. Gimon^a and Clifford V. Johnson^b

^a*Department of Physics /* ^b*Institute for Theoretical Physics,*
University of California,
Santa Barbara, CA 93106, USA

Abstract

We study string theory propagating on $\mathbf{R}^6 \times K3$ by constructing orientifolds of Type IIB string theory compactified on the T^4/\mathbf{Z}_N orbifold limits of the $K3$ surface. This generalises the \mathbf{Z}_2 case studied previously. The orientifold models studied may be divided into two broad categories, sometimes related by T-duality. Models in category *A* require either both D5- and D9-branes, or only D9-branes, for consistency. Models in category *B* require either only D5-branes, or no D-branes at all. This latter case is an example of a consistent purely closed unoriented string theory. The spectra of the resulting six dimensional $\mathcal{N}=1$ supergravity theories are presented. Precise statements are made about the relation of the \mathbf{Z}_N ALE spaces and D5-branes to instantons in the dual heterotic string theory.

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^aegimon@physics.ucsb.edu, ^bcvj@itp.ucsb.edu

1. Introduction

A significant advance in the ability of string theorists to calculate in both perturbative and non-perturbative string theory was made when the link between D-branes[1] and Ramond–Ramond (R-R) states in types I and II string theory was established[2]. The precise statement is that D-branes, extended objects with $(p + 1)$ -dimensional world volume¹ are the natural carriers of the fundamental units of $(p + 2)$ -form R-R field strength charge. Furthermore, D-branes are Bogomol’nyi–Prasad–Sommerfeld (BPS) saturated states, the very states about which the most far-reaching statements in non-perturbative string theory have been made. The traditional string theory toolbox is well equipped to study many aspects of D-branes². Consequently, many of the conjectures and partial computations which were made concerning non-perturbative string theory, (including suggestions that other extended objects were unavoidable outside perturbation theory[4,5,6]) were able to be tested and improved using D-brane technology.

Of the recent developments in string theory, ‘duality’ has been the physical phenomenon receiving most of the attention, due to the wide range of results and ideas it organises. One of the most striking manifestations of duality has been the idea that a string theory in its strong coupling limit yields another — weakly coupled — string theory[7][8]. Specifically, each of the known superstring theories have been demonstrated (sometimes after a compactification) to be related to another by some type of strong–weak coupling duality. A result of this has been that the previous emphasis (motivated by phenomenology) on heterotic string theory has become less pronounced: It is now believed that the known superstring theories are different perturbative realisations of some more profound underlying theory. In order to understand the nature of this unknown theory it seems prudent to gather as much data about it as possible, by studying more examples of stringy vacua which will give us insight into duality and related issues.

One such string–string duality pair is given by the $SO(32)$ ‘type I’ and ‘heterotic’ string theories. As ten dimensional theories, they were conjectured to be strong–weak coupling duals in ref.[8]. The ideas presented there, together with the later computations in refs.[9,10], were lent considerable support by the convincing computations (involving D-branes) carried out in ref.[11]. A natural step in studying this duality family further is to study non-trivial compactifications of both theories which are dual to one another. An immediate consequence of such a study was the realisation that there can be non-perturbative contributions to the gauge group of $SO(32)$ heterotic string theory (a phenomenon previously noted only in type II string theory[8][6]), which have a perturbative origin in the type I theory[12]. The observed non-perturbative phenomena were identified with the zero-size limit of heterotic instantons, which are ordinary solitonic fivebranes[13] in heterotic string

¹ We shall hereafter refer specifically to the p -dimensional objects as ‘D p -branes’. This will allow us to reserve terms like ‘D-brane’ for the generic case.

² For a review, see ref.[3].

theory (i.e., they give order e^{-1/λ_h^2} effects) and Dirichlet fivebranes (order e^{-1/λ_I}) in the type I theory. In the type I theory, these fivebranes are groups of D5-branes which are constrained to move as a single unit. The simplest such object is a pair of D5-branes, carrying $SU(2)$ Chan–Paton factors, as conjectured in ref.[12]. An immediate consequence of this conjecture was that a family of $2M$ D5-branes has gauge group $USp(2M)$. This corresponds to a non-perturbative contribution to the gauge group of the heterotic string, where it arises as M small instantons coincide.

Further investigations into such dual phenomena, and involving other compactifications of the dual theories, require more understanding of perturbative type I compactifications. The technology of such compactifications is perhaps the least developed of all of the superstring theories, probably due to the fact that open and unoriented string theories are not fully constrained by modular invariance of the worldsheet theory, in contrast to the closed oriented cases. The consistency of such compactifications traditionally involves computations which may be interpreted as the cancelation of tadpoles[14,15] and the removal of spacetime anomalies. This is still true at the present time.

The extension of the consistency techniques to include more general D-brane configurations was presented quite recently in ref.[16]. A perturbative type I derivation of the aforementioned symplectic projection in the D5-brane sector was presented there, confirming the conjecture and subsequent results of ref.[12]. In addition, ref.[16] presented an example of a non-trivial compactification: The type I string on $K3$ in its \mathbf{Z}_2 orbifold limit. This example is interesting for a number of reasons. One of these is the appearance of a new mechanism for the appearance of enhanced gauge symmetry groups: The basic dynamical fivebrane in the models of [16] is composed of four D5-branes, contributing $SU(2)$ to the gauge group when in isolation and away from the fixed points of the orbifold. The number four arises because the \mathbf{Z}_2 action on flat spacetime is realised on at least two D5-branes, while the orientation projection (by which we obtain type I strings from type II strings) requires a further doubling. We can therefore think of the four-D5-brane object in this case as two (mirror image) copies of the two-D5-brane unit discussed in ref.[8].

In the models of ref.[16], the enhancement of the gauge symmetry to $USp(2M)$ when M of these dynamical fivebranes coincide occurs in the same way as in the ten dimensional theory. However there is extra structure in the $K3(\mathbf{Z}_2)$ case, due to the presence of the sixteen orbifold fixed points. When $2M$ D5-branes coincide at one of these fixed points, there is an enhanced gauge symmetry $U(M)$, an interesting new feature. Another interesting observation is the fact that some of the models arising in this type I $K3(\mathbf{Z}_2)$ compactification have interesting heterotic duals[17]. The gauge group has a contribution from D9-branes and D5-branes. Generalising the results of [12], on the heterotic side the origin of this gauge group is therefore viewed as having both perturbative and non-perturbative components, in the dual heterotic string theory. The details of this have been worked out in ref.[17]. Also discussed there is the interesting fact that the spectra which arise have been shown to have a new class of $U(1)$ anomalies for which the anomaly polynomial does

not factorise. These anomalies can be shown to be canceled by a generalisation of the standard Green–Schwarz mechanism[18][19][17][20].

In this paper we present further studies of the type I $K3$ compactifications, considering other orbifold limits³. Specifically, our focus here will be to apply the consistency constraints presented in ref.[16], in the case where the orientifold group (with which we construct type I theories from type IIB) contains \mathbf{Z}_N as its spacetime symmetry group. The models arising are a natural extension of the models of [16]. One might expect that they will shed further light upon the nature of the types of symmetry enhancement possible at the orbifold points of the $K3(\mathbf{Z}_N)$ models, and this is indeed the case. For $K3(\mathbf{Z}_N)$ the basic dynamical unit is now $2N$ D5–branes; N for a natural action of \mathbf{Z}_N in spacetime on the D5–branes, together with the pairing in the orientifold. The coincidence of these dynamical fivebranes results in symplectic gauge groups as before, with enhancement to unitary groups at the orbifold fixed points. Particularly interesting is the fact that for \mathbf{Z}_4 and \mathbf{Z}_6 , the types of fixed points are mixed, which makes for extra non–trivial structure which helps us to deduce certain facts about the nature of open string theory near the \mathbf{Z}_N ALE spaces in their blow–down limit.

After a brief review in section 2 of the properties of $K3$ ’s \mathbf{Z}_N orbifold limits which we will use, we study in section 3 how the action of \mathbf{Z}_N orientifold groups are realised at the level of Chan–Paton factors. In section 4, we present detailed computation of the twisted sector tadpoles which arise in the neighbourhood of the fixed points of an orbifold. This supplies us with useful information about the nature of type I string theory on the zero size limit of \mathbf{Z}_N ALE instantons, out of which a $K3$ manifold may be constructed in the standard way. In section 5 we put all this information together and compute the complete spectra of type IIB string theory on $K3$ orientifolds. Throughout, we discuss and develop results as they arise, concluding in section 6 with a brief summary and discussion. The data we gained about these models will have a lot to teach us about the dual heterotic string theory, and we briefly interpret our results in this light.

2. ALE Spaces, Orbifolds and $K3$

We will study string propagation on $\mathbf{R}^6 \times K3$, denoting by X^μ , $\mu = 0, 1, \dots, 5$, the non–compact coordinates and X^m , $m = 6, \dots, 9$ the compact coordinates. For most of the presentation we will consider $K3$ in its orbifold limits, T^4/\mathbf{Z}_N . To construct the orbifold T^4/\mathbf{Z}_N , we begin with the space $\mathbf{R}^4 \equiv \mathbf{C}^2$, with complex coordinates $z^1 = X^6 + iX^7$ and $z^2 = X^8 + iX^9$, upon which we make the identifications $z^i \sim z^i + 1 \sim z^i + i$, for $N=2$ or 4, and $z^i \sim z^i + 1 \sim z^i + \exp(\pi i/3)$ for $N=3$ or 6. These lattices define for us the torus T^4 , upon which the discrete rotations \mathbf{Z}_N , acts naturally as $(z^1, z^2) \rightarrow (\beta z^1, \beta^{-1} z^2)$, for $\beta = \exp(2\pi i/N)$.

³ See also ref.[21] for more work in this context.

We may therefore define a new space by identifying points under the action of \mathbf{Z}_N . This is the orbifold T^4/\mathbf{Z}_N , which is a smooth surface except at points at which the curvature of the orbifold is located. These ‘fixed points’ are points which are invariant under \mathbf{Z}_N or some non-trivial subgroup of it. For $N \in \{2, 3, 4, 6\}$, this procedure produces a family of compact spaces which are the orbifold limits of the $K3$ surface.

The smooth $K3$ manifold is obtained from these limits by ‘blowing up’ the orbifold points. This procedure is simply the process by which each of the points is removed and replaced by a smooth space. For the orbifold T^4/\mathbf{Z}_N the neighbourhood of a fixed point is $\mathbf{R}^4/\mathbf{Z}_M$, where $N \geq M \in \{2, 3, 4, 6\}$. This is the asymptotic region of the A-series ALE gravitational instanton[22,23] (denoted here \mathcal{E}_M), with which we replace the excised point.

Denote the generator of \mathbf{Z}_N by α_N , the group elements being the powers α_N^m , for $m \in \{0, 1, \dots, N-1\}$. First, we observe that the number, F_M , of points fixed under the \mathbf{Z}_M subgroup of \mathbf{Z}_N , (generated by $\alpha_N^{N/M}$) is simply $F_M = 16 \sin^4 \frac{\pi}{M}$, where M is a multiple of N .

Let us review the fixed point structure of each space. For more details, see refs.[24,25]. For T^4/\mathbf{Z}_2 we have 16 points fixed under the action of α_2 , each of which are replaced by the space \mathcal{E}_2 , in order to resolve to smooth $K3$. Meanwhile for T^4/\mathbf{Z}_3 there are 9 fixed points of α_3 , which are each replaced by \mathcal{E}_3 in the blow-up.

The case T^4/\mathbf{Z}_4 has 16 fixed points. Four of them are fixed under the action of α_4 , while the other 12 are only fixed under α_4^2 . Under α_4 , these 12 \mathbf{Z}_2 points transform as 6 doublets. Consequently, the blow-up is carried out by first constructing the \mathbf{Z}_4 -invariant region by identifying these pairs of fixed points, and then replacing each of the original 4 \mathbf{Z}_4 fixed points by an \mathcal{E}_4 and the 6 pairs by an \mathcal{E}_2 .

For T^4/\mathbf{Z}_6 the situation is similar. There are 24 fixed points altogether. There is only one point fixed under α_6 . It is replaced by \mathcal{E}_6 in the blow-up. There are 8 points fixed under the \mathbf{Z}_3 subgroup, generated by α_6^2 , which transform as doublets under the action of α_6 . They are therefore replaced by 4 copies of \mathcal{E}_3 . There are 15 points fixed under α_6^3 , which transform as triplets under the action of α_6 . Consequently, they are replaced by 5 copies of \mathcal{E}_2 in performing the blow-up surgery.

Knowledge of the properties of the ALE spaces \mathcal{E}_m which are used in the blow-up procedure can be used to show that many of the properties of $K3$ can be deduced from this construction[24].

3. Orientifolds and Chan–Paton Factors

We wish to consider type I string theory propagating on the spaces we described above. Such theories are constructed here by orientifolding type IIB string theory and introducing open string boundary conditions. An orientifold generalises the concept of an orbifold

to include not only spacetime symmetries, but also world sheet parity symmetry in the group of discrete symmetries which are gauged. The introduction of open string sectors is somewhat analogous to introducing twisted sectors in an orbifold theory[26].

Our spacetime symmetry group is \mathbf{Z}_N . As before, we denote the generator of this group by α_N , the group elements being the powers α_N^k , for $k \in \{0, 1, 2, \dots, N-1\}$. The orientifold group can therefore contain the elements α_N^k and also $\Omega \cdot \alpha_N^k$ (which we shall sometimes denote Ω_k), where Ω is world sheet parity. Gauging the action of α_N^k will introduce the familiar closed string twisted sectors for an orbifold, while gauging Ω_k will result in unoriented surfaces in the world-sheet expansion.

We have a choice as to the elements which we wish to consider in our orientifold group, our only constraint being closure, of course. Let us denote the two choices of \mathbf{Z}_N orientifold group as \mathbf{Z}_N^A and \mathbf{Z}_N^B . The choice analogous to ref.[16] is to have

$$\mathbf{Z}_N^A = \{1, \Omega, \alpha_N^k, \Omega_j\}, \quad k, j = 1, 2, \dots, N-1. \quad (3.1)$$

A second choice (only for N even) is

$$\mathbf{Z}_N^B = \{1, \alpha_N^{2k-2}, \Omega_{2j-1}\}, \quad k, j = 1, 2, \dots, \frac{N}{2}. \quad (3.2)$$

Both of these orientifold groups are consistent, as they both close under group multiplication. We shall see the consequences of each choice of orientifold group as we proceed.

Consistency will require the addition of open string sectors[26]. Open string endpoints will lie on various submanifolds of spacetime, the D-branes. Each such submanifold is labeled by a Chan-Paton index corresponding to the state of an open string endpoint. An open string state will be denoted $|\psi, ij\rangle$, where ψ is the state of the world-sheet fields and i and j are the states of the string end-points. For a consistent construction, we must determine how the action of the orientifold group is manifested at the level of Chan-Paton factors. In general, for every D-brane which exists in the theory, the spacetime transformed D-brane must also appear[16]. The action of an orbifold group element g on this complete set will be represented by the unitary matrices γ_g which act on the open string endpoints.

Constraints on the γ 's arise when we consider the action of various orientifold symmetries: They must form a faithful representation of the group, up to phases. We have for example

$$\alpha_N^k : |\psi, ij\rangle \rightarrow (\gamma_k)_{ii'} |\alpha_N^k \cdot \psi, i'j'\rangle (\gamma_k^{-1})_{j'j} \quad (3.3)$$

while for $\Omega \cdot \alpha_N^k \equiv \Omega_k$,

$$\Omega_k : |\psi, ij\rangle \rightarrow (\gamma_{\Omega_k})_{ii'} |\Omega_k \cdot \psi, j'i'\rangle (\gamma_{\Omega_k}^{-1})_{j'j}. \quad (3.4)$$

Notice that when that action includes Ω , the ends of the string are transposed. Composing various actions of the group elements, we see that as $(\alpha_N^k)^N = 1$, then

$$(\alpha_N^k)^N : |\psi, ij\rangle \rightarrow (\gamma_k^N)_{ii'} |\psi, i'j'\rangle (\gamma_k^{-N})_{j'j} \quad (3.5)$$

and so

$$\gamma_k^N = \pm 1. \quad (3.6)$$

Similarly, as $\Omega^2 = 1$

$$\Omega^2 : |\psi, ij\rangle \rightarrow (\gamma_\Omega (\gamma_\Omega^T)^{-1})_{ii'} |\psi, i'j'\rangle (\gamma_\Omega^T \gamma_\Omega^{-1})_{j'j}, \quad (3.7)$$

resulting in

$$\gamma_\Omega = \pm \gamma_\Omega^T. \quad (3.8)$$

Other examples of such conditions will be put to explicit use later when solving the tadpole equations. Another important example of a physical constraint is the ‘superselection rule’ which implies that there should be no non-zero elements of the Chan–Paton matrices, λ , which connect D-branes which are at different points in spacetime.

4. Tadpoles for ALE \mathbf{Z}_M Singularities.

4.1. Consistent Field Equations

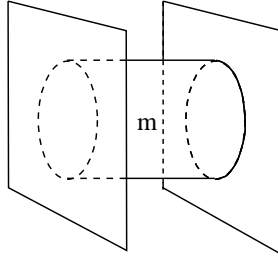
In open and/or unoriented string theory, certain divergences arise at the one-loop level, which may be interpreted[14] as inconsistencies in the field equations for the R-R potentials in the theory. They manifest themselves as tadpoles. A minimum requirement in the construction of a consistent theory is that we ensure that these tadpoles are all canceled. These tadpoles are topologically of two types, disc tadpoles and \mathbf{RP}^2 tadpoles. They are perhaps best visualised as the process of emitting an R-R closed string state from a Dp -brane (for the disc), a source of $(p+1)$ -form R-R potential, or from an orientifold plane (for \mathbf{RP}^2), which also carries R-R charge. The prototype example of the cancelation of these tadpoles is the ten dimensional $SO(32)$ open (type I) string theory. In that case the disc and \mathbf{RP}^2 produce a divergence proportional to $(n_9-32)^2$ for $SO(n_9)$ Chan–Paton factors (i.e., there are n_9 D9-branes[2]) and $(n_9+32)^2$ for $USp(n_9)$. Cancelation of the divergences therefore requires gauge group $SO(32)$ (i.e., 32 D9-branes). In that case, the orientifold group was very simple, the only element being the purely internal Ω , and thus there are no spacetime symmetries to consider. The tadpole cancelation there was for global consistency of the 10-form potential’s field equation.

In our case, we have all of the spacetime symmetries \mathbf{Z}_N to include. So in general, we will have tadpole contributions from not only the familiar R-R potentials, but also twisted R-R potentials. For orientifold group \mathbf{Z}_N^A , which contains Ω , we will have D9-branes present, as in ten dimensional type I string theory and in the \mathbf{Z}_2 example of ref.[16]. In the case of \mathbf{Z}_N^B , there is no element Ω , and we will have no requirement to include D9-branes, as in the example of [20]. In the case of type A orientifolds we will always have D5-branes whenever N is even. This is simply because only for N even does there exist a \mathbf{Z}_2 subgroup

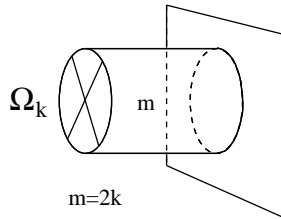
of \mathbf{Z}_N . There are many ways to see that this results in D5-branes. It will be particularly obvious when we consider the tadpole diagrams later.

One way to think about it is to realise that the \mathbf{Z}_2 subgroup acts as reflections in the X^m , ($m = 6, \dots, 9$) directions, generated by $\alpha_N^{N/2}$ (called R in ref.[16]). Therefore there is also the element $\Omega_{N/2} \equiv \Omega R$ in the orientifold group. Under T -duality in the directions in which R acts, ΩR becomes Ω . Whatever the details of the dual model, it must contain D9-branes, because of the presence of Ω . T -duality exchanges Dirichlet (D) and Neumann (N) boundary conditions. Therefore, the original model with orientifold group containing the element ΩR contains D5-branes, with X^m as their Dirichlet coordinates, their world-volumes filling the non-compact \mathbf{R}^6 .

Henceforth, we shall use \mathbf{Z}_M for the subgroups of the \mathbf{Z}_N we use for the orbifold, where $M \leq N$ is a factor of N . This will be used to denote the generic subgroup under which the orbifold singularities (with local ALE geometry $\mathbf{R}^4/\mathbf{Z}_M$) are fixed. (N will be reserved for the parent group). D-branes and fixed points are sources for R-R field strengths. In particular, a D5-brane sitting at a \mathbf{Z}_M fixed point can be a source for various α_M^m -twisted 6-form R-R potentials, A_6^m , with field strength H_7^m , where $m = 1, 2, \dots, M-1$. The open string one-loop diagram corresponding to the twisted cylinder shows two D5-branes exchanging such a form in the closed string channel:



Let us denote the charge of the D5-brane by μ_5^m . A crosscap diagram arises in the neighbourhood of a fixed point, which is also an orientifold point. The orientifold/fixed point identifies strings under Ω together with a phase from the group element α_M^k . This is the orientifold group element Ω_k : fixed points can have charge under the A_6^m , when $m = 2k$. The open string amplitude depicting the interaction between a D5-brane the fixed point is a Möbius strip:



From this amplitude, one may formally extract the value of the force between the D5-brane

and the fixed point, as in ref.[3]. By comparing it to the force between two D5-branes, obtained from the cylinder amplitude we may deduce the charge $\tilde{\mu}_5^m$ of the fixed point. Note, however that for non-trivial twists by m , the string zero modes vanish, and therefore the fivebrane is forced to sit at the fixed point. This simply means that the twisted 6-form R-R potentials A_6^m cannot propagate in spacetime. Therefore, their equations of motion must be satisfied locally. It is only when we consider the untwisted R-R six-form field strength A_6 (i.e., $m=0 \rightarrow k=M/2 \bmod M$) that we see how to separate the D5-branes from the fixed points. As we shall see, there will be freedom in the final equations to group together a collection of D5-branes, forming them into a collective object with H_7^m charge $\mu_5^m = 0$, but non-zero charge μ_5 under the untwisted field strength, H_7 . This fivebrane is now free to move away from fixed point; it is the generalisation of the dynamical fivebrane of refs.[12] and [16]. In the case of ref.[12], the fivebrane (in the ten dimensional theory) was composed of two D5-branes, while on $K3(\mathbf{Z}_2)$ it was composed of four of them[16]. In the present $K3(\mathbf{Z}_N)$ context, we shall see that the dynamical fivebrane is composed of $2N$ D5-branes (for orientifold group of type A). As the fivebrane, charged under H_7 , can move about in the internal space, the conservation equations for A_6 charge need not be canceled locally. Flux lines can extend throughout the compact internal space, the only requirement being global charge conservation. In the case when the orientifold group is \mathbf{Z}_N^A , we also have the familiar ten-form potential's field equations to satisfy globally.

The way which we shall proceed is as follows. We shall compute all of the tadpoles arising from the various orientifold group elements which might appear, focusing our attention initially on the twisted 6-form potentials' tadpoles. We can therefore carry out a local analysis, in the neighbourhood of a fixed point. Therefore, we can simply consider our 'internal' space to be $\mathbf{R}^4/\mathbf{Z}_N$, (the zero-size 'blow-down' limit of an ALE instanton). Our 'internal' space is therefore non-compact, for the purposes of this local computation. We can therefore ignore most of the clutter of the propagation of zero modes and the presence of winding modes in the 'internal' directions, as these will not be relevant in the limit. The tadpoles which arise from the untwisted 6-form and 10-form potentials will be identical to those already computed in ref.[16]. (We will nevertheless compute and present them here, for completeness.)

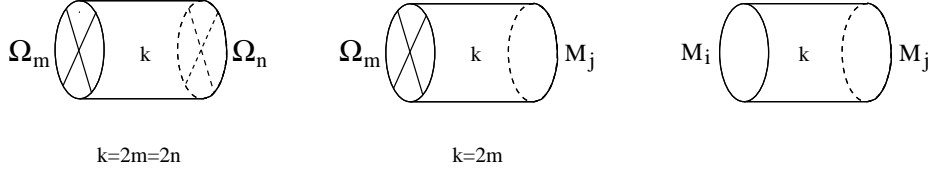
The information we will gather about local tadpole cancelation at an ALE point can then be used to construct the complete $K3$ orbifold model, by using the knowledge we have about how $K3$ is constructed from such points (reviewed in section 2).

4.2. Preliminaries

The most efficient way of computing the divergent contribution of the tadpoles is to compute the one-loop diagrams (the Klein bottle (KB), Möbius strip (MS) and cylinder (C)) and then to take a limit which extracts the divergent pieces. The fact that these diagrams yield the disc and \mathbf{RP}^2 tadpoles in terms of the sums of three different products means

that the requirement of factorisation of the final expression is a strong consistency check on the whole computation.

The three consistent types of diagram which can be drawn, labeled by the possible elements of the orientifold group under consideration are depicted below:



In the figure, the crosscaps show the action of Ω_m as one goes half way around the open string channel (*around* the cylinder). Going around all the way picks up another action of Ω_m , yielding the \mathbf{Z}_N element $\Omega_m^2 = \alpha_N^{2m}$ which is the twist which propagates in the closed string channel (*along* the cylinder). For consistency, if there is a crosscap with Ω_n at the other end, forming a Klein bottle, then $\Omega_n^2 = \alpha_N^{2n}$ should yield the same twist in the closed string channel, *i.e.*, $2n = 2m$. For \mathbf{Z}_N with N odd, there is only one solution to this: $m = n \bmod N$. When N is even however, we can have also the solution $n = m + N/2 \bmod N$. Note also that in the figure, M_i denotes the manifold i , upon which an open string can end: a D-brane.

Let us parameterise the surfaces as cylinders with length $2\pi l$ and circumference 2π with either boundaries or crosscaps on their ends with boundary conditions on a generic field ϕ (and its derivatives):

$$\begin{aligned}
\text{KB : } & \phi(0, \pi + \sigma^2) = \Omega_m \cdot \phi(0, \sigma^2) \\
& \phi(2\pi l, \pi + \sigma^2) = \Omega_n \cdot \phi(2\pi l, \sigma^2) \\
& \phi(\sigma^1, 2\pi + \sigma^2) = \alpha_N^k \cdot \phi(\sigma^1, \sigma^2); \quad k = 2m = 2n \\
\text{MS : } & \phi(2\pi l, \sigma^2) \in M_j \\
& \phi(0, \pi + \sigma^2) = \Omega_m \cdot \phi(0, \sigma^2) \\
& \phi(\sigma^1, 2\pi + \sigma^2) = \alpha_N^k \cdot \phi(\sigma^1, \sigma^2); \quad k = 2m \\
\text{C : } & \phi(0, \sigma^2) \in M_i \\
& \phi(2\pi l, \pi + \sigma^2) \in M_j \\
& \phi(\sigma^1, 2\pi + \sigma^2) = \alpha_N^k \cdot \phi(\sigma^1, \sigma^2)
\end{aligned} \tag{4.1}$$

In computing the traces to yield the one-loop expressions, it is convenient to parameterise

the Klein bottle and Möbius strip in the region $0 \leq \sigma^1 \leq 4\pi l, 0 \leq \sigma^2 \leq \pi$ as follows:

$$\begin{aligned}
\text{KB : } \quad & \phi(\sigma^1, \pi + \sigma^2) = \Omega_m \cdot \phi(4\pi l - \sigma^1, \sigma^2) \\
& \phi(4\pi l, \sigma^2) = \alpha_N^{m-n} \cdot \phi(0, \sigma^2) \\
\text{MS : } \quad & \phi(0, \sigma^2) \in M_j \\
& \phi(4\pi l, \sigma^2) \in M_j \\
& \phi(\sigma^1, \pi + \sigma^2) = \Omega_m \cdot \phi(4\pi l - \sigma^1, \sigma^2).
\end{aligned} \tag{4.2}$$

After the standard rescaling of the coordinates such that open strings are length π while closed strings are length 2π , the amplitudes are

$$\begin{aligned}
\text{KB : } \quad & \text{Tr}_{c,k} \left(\Omega_m (-1)^{F+\tilde{F}} e^{\pi(L_0+\bar{L}_0)/2l} \right) \\
\text{MS : } \quad & \text{Tr}_{o,jj} \left(\Omega_m (-1)^F e^{\pi L_0/4l} \right) \\
\text{C : } \quad & \text{Tr}_{o,ij} \left(\alpha_N^k (-1)^F e^{\pi L_0/l} \right).
\end{aligned} \tag{4.3}$$

(Here ‘ o ’ and ‘ c ’ mean ‘open’ and ‘closed’, respectively.)

The complete one-loop amplitude is

$$\int_0^\infty \frac{dt}{t} \left\{ \text{Tr}_c \left(\mathbf{P} (-1)^{\mathbf{F}} e^{-2\pi t(L_0+\bar{L}_0)} \right) + \text{Tr}_o \left(\mathbf{P} (-1)^{\mathbf{F}} e^{-2\pi t L_0} \right) \right\}. \tag{4.4}$$

The projector \mathbf{P} includes the GSO and group projections and \mathbf{F} is the spacetime fermion number. The traces are over transverse oscillator states and include sums over spacetime momenta. After we evaluate the traces, the $t \rightarrow 0$ limit will yield the divergences. Note also that the loop modulus t is related to the cylinder length l as $t = 1/4l, 1/8l$ and $1/2l$ for the Klein bottle, Möbius strip and cylinder, respectively.

In order to compute the loop amplitudes, we must first decide upon a consistent convention for the action of \mathbf{Z}_N and Ω on the various sectors of the theory. We follow most of the conventions used in ref.[16]. In addition to the conventions listed there, we have that the elements α_N^k act as follows on the bosons and in the Neveu–Schwarz (NS) sector:

$$\alpha_N^k : \begin{cases} z_1 = X^6 + iX^7 \rightarrow e^{\frac{2\pi ik}{N}} z_1, \\ z_2 = X^8 + iX^9 \rightarrow e^{-\frac{2\pi ik}{N}} z_2 \end{cases}, \tag{4.5}$$

and it acts in the Ramond (R) sector as

$$\alpha_N^k = e^{\frac{2\pi ik}{N}(J_{67}-J_{89})}. \tag{4.6}$$

As a consequence of this latter convention (which has an extra relative minus sign relative to the analogous operator in ref.[16]), notice for example that α_N^k gives $4\cos^2 \frac{\pi k}{N}$ when evaluated on the R ground states while $(-)^F \alpha_N^k$ gives $4\sin^2 \frac{\pi k}{N}$.

4.3. Loop amplitudes

Explicitly, we compute the following amplitudes, generalising[16]:

$$\begin{aligned}
\text{KB : } & \text{Tr}_{NSNS+RR}^{U+T} \left\{ \frac{\Omega}{2} \sum_{k=0}^{N-1} \frac{\alpha_N^k}{N} \cdot \frac{1 + (-1)^F}{2} \cdot e^{-2\pi t(L_0 + \bar{L}_0)} \right\} \\
\text{MS : } & \text{Tr}_{NS-R}^{99+55} \left\{ \frac{\Omega}{2} \sum_{k=0}^{N-1} \frac{\alpha_N^k}{N} \cdot \frac{1 + (-1)^F}{2} \cdot e^{-2\pi t L_0} \right\} \\
\text{C : } & \text{Tr}_{NS-R}^{99+55+95+59} \left\{ \frac{1}{2} \sum_{k=0}^{N-1} \frac{\alpha_N^k}{N} \cdot \frac{1 + (-1)^F}{2} \cdot e^{-2\pi t L_0} \right\},
\end{aligned} \tag{4.7}$$

where $U(T)$ refers to the untwisted (twisted) sector of the closed string. As Ω forces the left and right moving sector to be identical, there is no need to include $\frac{1}{2}(1 + (-1)^{\bar{F}})$ in the trace in the Klein bottle. The open string traces include a sum over Chan–Paton factors.

Before listing the results of the careful computations, which yield the one-loop amplitudes, let us introduce some of the characters and notation which will appear.

Playing a central role in organising the amplitudes will be Jacobi’s ϑ -functions:

$$\begin{aligned}
\vartheta_1(z|t) &= 2q^{1/4} \sin \pi z \prod_{n=1}^{\infty} (1 - q^{2n}) \prod_{n=1}^{\infty} (1 - q^{2n} e^{2\pi i z}) \prod_{n=1}^{\infty} (1 - q^{2n} e^{-2\pi i z}) \\
\vartheta_2(z|t) &= 2q^{1/4} \cos \pi z \prod_{n=1}^{\infty} (1 - q^{2n}) \prod_{n=1}^{\infty} (1 + q^{2n} e^{2\pi i z}) \prod_{n=1}^{\infty} (1 + q^{2n} e^{-2\pi i z}) \\
\vartheta_3(z|t) &= \prod_{n=1}^{\infty} (1 - q^{2n}) \prod_{n=1}^{\infty} (1 + q^{2n-1} e^{2\pi i z}) \prod_{n=1}^{\infty} (1 + q^{2n-1} e^{-2\pi i z}) \\
\vartheta_4(z|t) &= \prod_{n=1}^{\infty} (1 - q^{2n}) \prod_{n=1}^{\infty} (1 - q^{2n-1} e^{2\pi i z}) \prod_{n=1}^{\infty} (1 - q^{2n-1} e^{-2\pi i z})
\end{aligned} \tag{4.8}$$

where $q = e^{-\pi t}$. We will need their asymptotics at $t \rightarrow 0$. The asymptotics as $t \rightarrow \infty$ are straightforward. The asymptotics as $t \rightarrow 0$ are obtained from the modular transformations ($\tau = it$)

$$\begin{aligned}
\vartheta_1(z|\tau) &= \tau^{-1/2} e^{3i\pi/4} e^{-i\pi z^2/\tau} \vartheta_1\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}\right), & \vartheta_3(z|\tau) &= \tau^{-1/2} e^{i\pi/4} e^{-i\pi z^2/\tau} \vartheta_3\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}\right), \\
\vartheta_2(z|\tau) &= \tau^{-1/2} e^{i\pi/4} e^{-i\pi z^2/\tau} \vartheta_4\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}\right), & \vartheta_4(z|\tau) &= \tau^{-1/2} e^{i\pi/4} e^{-i\pi z^2/\tau} \vartheta_2\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}\right).
\end{aligned} \tag{4.9}$$

The f -functions familiar from ten dimensional string theory are a special case of the

functions (4.8) (here, a prime denotes $\partial/\partial z$):

$$\begin{aligned}
f_1(q) &= q^{1/12} \prod_{n=1}^{\infty} (1 - q^{2n}) = (2\pi)^{-1/3} \vartheta'_1(0|t)^{1/3} \\
f_2(q) &= \sqrt{2} q^{1/12} \prod_{n=1}^{\infty} (1 + q^{2n}) = (2\pi)^{1/6} \vartheta'_2(0|t)^{1/2} \vartheta'_1(0|t)^{-1/6} \\
f_3(q) &= q^{-1/24} \prod_{n=1}^{\infty} (1 + q^{2n-1}) = (2\pi)^{1/6} \vartheta'_3(0|t)^{1/2} \vartheta'_1(0|t)^{-1/6} \\
f_4(q) &= q^{-1/24} \prod_{n=1}^{\infty} (1 - q^{2n-1}) = (2\pi)^{1/6} \vartheta'_4(0|t)^{1/2} \vartheta'_1(0|t)^{-1/6}
\end{aligned} \tag{4.10}$$

Their asymptotics at $t \rightarrow 0$ follow from their modular transformations and together with their $t \rightarrow \infty$ behaviour:

$$f_1(e^{-\pi/s}) = \sqrt{s} f_1(e^{-\pi s}), \quad f_3(e^{-\pi/s}) = f_3(e^{-\pi s}), \quad f_2(e^{-\pi/s}) = f_4(e^{-\pi s}). \tag{4.11}$$

The familiar “*aequatio identico satis abstrusa*”

$$f_3(q)^8 - f_4(q)^8 - f_2^8(q) = 0, \tag{4.12}$$

follows from the more general identities

$$\begin{aligned}
\vartheta_3^2(0|t) \vartheta_3^2(z|t) - \vartheta_4^2(0|t) \vartheta_4^2(z|t) - \vartheta_2^2(0|t) \vartheta_2^2(z|t) &= 0 \\
\vartheta_3^2(0|t) \vartheta_2^2(z|t) - \vartheta_4^2(0|t) \vartheta_1^2(z|t) - \vartheta_2^2(0|t) \vartheta_3^2(z|t) &= 0 \\
\vartheta_3^2(0|t) \vartheta_4^2(z|t) - \vartheta_4^2(0|t) \vartheta_3^2(z|t) - \vartheta_2^2(0|t) \vartheta_1^2(z|t) &= 0,
\end{aligned} \tag{4.13}$$

of which we will make much use in what is to follow.

The appearance of the full Jacobi ϑ -functions is, in retrospect, perhaps not surprising, as twisting in the open string loop channel by α_N^k introduces $z = k/N$ into the oscillator sums. They therefore arise naturally in the cylinder and Möbius strip amplitudes listed below. In the case of the Klein bottle, there is also a twist in the closed string loop channel by α_N^{n-m} . Such a space twist will in general change the moding of the fermion and bosons, producing a ‘spectral flow’ between all of the different sectors. This should manifest itself as another type of twist of the ϑ -functions. To write this relationship, we use the notation

$$\vartheta_1 = \vartheta\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right], \quad \vartheta_2 = \vartheta\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \quad \vartheta_3 = \vartheta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \quad \vartheta_4 = \vartheta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \tag{4.14}$$

in which we can succinctly write[27]:

$$\vartheta\left[\begin{smallmatrix} \epsilon \\ \epsilon' \end{smallmatrix}\right](z - \zeta t|t) = e^{i\pi\{-\tau\zeta^2 + \zeta\epsilon' + 2\zeta z\}} \vartheta\left[\begin{smallmatrix} \epsilon - 2\zeta \\ \epsilon' \end{smallmatrix}\right](z|t), \tag{4.15}$$

where $(\epsilon, \epsilon') \in \{0, 1\}$ for the familiar ϑ -functions. In evaluating the Klein bottle amplitude, the relations (4.15) are used to rewrite twisted expressions in terms of ϑ -functions.

For the twisted 99 cylinders the one-loop amplitudes are ($z=k/N$):

$$\begin{aligned} \frac{V_6}{2^3 N} \sum_{k=1}^{N-1} \frac{(\text{Tr}(\gamma_{k,9}))^2}{(4 \sin^2 \pi z)^2} \int_0^\infty \frac{dt}{t} (8\pi^2 \alpha' t)^{-3} 4 \sin^2 \pi z f_1^{-6}(t) \vartheta_1^{-2}(z|t) \times \\ \{ \vartheta_3^2(0|t) \vartheta_3^2(z|t) - \vartheta_4^2(0|t) \vartheta_4^2(z|t) - \vartheta_2^2(0|t) \vartheta_2^2(z|t) \}, \end{aligned} \quad (4.16)$$

while for the twisted 55 cylinders they are:

$$\begin{aligned} \frac{V_6}{2^3 N} \sum_{k=1}^{N-1} (\text{Tr}(\gamma_{k,5}))^2 \int_0^\infty \frac{dt}{t} (8\pi^2 \alpha' t)^{-3} 4 \sin^2 \pi z f_1^{-6}(t) \vartheta_1^{-2}(z|t) \times \\ \{ \vartheta_3^2(0|t) \vartheta_3^2(z|t) - \vartheta_4^2(0|t) \vartheta_4^2(z|t) - \vartheta_2^2(0|t) \vartheta_2^2(z|t) \}. \end{aligned} \quad (4.17)$$

The 95 cylinders give:

$$\begin{aligned} 2 \frac{V_6}{2^3 N} \sum_{k=1}^{N-1} \text{Tr}(\gamma_{k,9}) \text{Tr}(\gamma_{k,5}) \int_0^\infty \frac{dt}{t} (8\pi^2 \alpha' t)^{-3} f_1^{-6}(t) \vartheta_4^{-2}(z|t) \times \\ \{ \vartheta_3^2(0|t) \vartheta_2^2(z|t) - \vartheta_4^2(0|t) \vartheta_1^2(z|t) - \vartheta_2^2(0|t) \vartheta_3^2(z|t) \}. \end{aligned} \quad (4.18)$$

The twisted Möbius strip amplitudes are, for the D5-branes ($z=m/N$):

$$\begin{aligned} -\frac{V_6}{2^3 N} \sum_{m=1}^{N-1} \text{Tr}(\gamma_{\Omega_m,5}^{-1} \gamma_{\Omega_m,5}^T) \int_0^\infty \frac{dt}{t} (8\pi^2 \alpha' t)^{-3} 4 \cos^2 \pi z f_1^{-6}(iq) \vartheta_2^{-2}(iq, z) \times \\ \{ \vartheta_3^2(iq, 0) \vartheta_4^2(iq, z) - \vartheta_4^2(iq, 0) \vartheta_3^2(iq, z) - \vartheta_2^2(iq, 0) \vartheta_1^2(iq, z) \}, \end{aligned} \quad (4.19)$$

and for the D9-branes:

$$\begin{aligned} -\frac{V_6}{2^3 N} \sum_{m=1}^{N-1} \frac{\text{Tr}(\gamma_{\Omega_m,9}^{-1} \gamma_{\Omega_m,9}^T)}{(4 \sin^2 \pi z)^2} \int_0^\infty \frac{dt}{t} (8\pi^2 \alpha' t)^{-3} 4 \sin^2 \pi z f_1^{-6}(iq) \vartheta_1^{-2}(iq, z) \times \\ \{ \vartheta_3^2(iq, 0) \vartheta_3^2(iq, z) - \vartheta_4^2(iq, 0) \vartheta_4^2(iq, z) - \vartheta_2^2(iq, 0) \vartheta_2^2(iq, z) \}. \end{aligned} \quad (4.20)$$

Finally, the Klein bottle gives ($t^+=t+\xi t$, $t^-=t-\xi t$):

$$\begin{aligned} \frac{V_6}{2^3 N} \sum_{m,n=1}^{N-1} \frac{1}{(4 \sin^2 \pi z)^2} \int_0^\infty \frac{dt}{t} (4\pi^2 \alpha' t)^{-3} 4 \sin^2 2\pi(z-\zeta t) f_1^{-6}(2t) \vartheta_1^{-1}(z|2t^-) \vartheta_1^{-1}(z|2t^+) \times \\ \{ -\vartheta_4^2(0|2t) \vartheta_4(z|2t^-) \vartheta_4(z|2t^+) + \vartheta_3^2(0|2t) \vartheta_3(z|2t^-) \vartheta_3(z|2t^+) \\ - \vartheta_2^2(0|2t) \vartheta_2(z|2t^-) \vartheta_2(z|2t^+) \}. \end{aligned} \quad (4.21)$$

In the Klein bottle amplitudes, we have the twist $\zeta=(m-n)/N$ in the closed string channel, resulting in a zero point energy shift for the bosons and fermions which contribute. V_6 is the regularised six dimensional spacetime volume.

The factor of $(4 \sin^2 \pi z)^{-2}$ is a non-trivial contribution from evaluating the trace of the operator \mathcal{O} in the z^1 and z^2 complex planes in the NN sector. The operator \mathcal{O} is the rotation

$$\mathcal{O} : \quad z^{1,2} \rightarrow e^{\pm \frac{2\pi i k}{N}} z^{1,2}. \quad (4.22)$$

We have

$$\text{Tr}[e^{\frac{2\pi i k}{N}}] = \int dz^1 dz^2 \langle z^1, z^2 | \mathcal{O} | z^{1'}, z^{2'} \rangle = \left(4 \sin^2 \frac{\pi k}{N} \right)^{-2}, \quad (4.23)$$

where we have used the basis

$$\langle z^1, z^2 | z^{1'}, z^{2'} \rangle = \frac{1}{V_{T^4}} \delta(z^1 - z^{1'}) \delta(z^2 - z^{2'}). \quad (4.24)$$

Supersymmetry is manifest here, as due to the identities (4.13) each of these amplitudes vanishes identically. However, we wish to extract the tadpoles for closed string massless NS-NS fields from these amplitudes, and we do so by identifying the contribution of this sector from each of these amplitudes.

4.4. Factorisation and Tadpoles

The next step is to extract the asymptotics as $t \rightarrow 0$ of the amplitudes, relating this limit to the $l \rightarrow \infty$ limit for each surface, (using the relation between l and t for each surface given earlier). Here, the asymptotic behaviour of the ϑ - and f -functions given in equations (4.9) and (4.11) are used. This extracts the (divergent) contribution of the massless closed string R-R fields, which we shall list below. In what follows, we shall neglect the overall factors of $1/N$ and powers of 2 which accompany all of the amplitudes.

First, we list the tadpoles for the untwisted R-R potentials. For the 10-form we have the following expression (proportional to $(1-1)v_6 v_4 \int_0^\infty dl$):

$$\text{Tr}(\gamma_{0,9})^2 - 64 \text{Tr}(\gamma_{\Omega,9}^{-1} \gamma_{\Omega,9}^T) + 32^2, \quad (4.25)$$

corresponding to the diagrams:

Here, $v_D = V_D (4\pi^2 \alpha')^{-D/2}$, where V_D is a regularised D dimensional volume. The limit where we focus upon the neighbourhood of one ALE point is equivalent to taking the non-compact limit $v_4 \rightarrow \infty$ while staying in a frame where our regulated volume $v_{10} = v_6 v_4$ is finite.

For the 6-form we have (proportional to $(1 - 1) \frac{v_6}{v_4} \int_0^\infty dl$):

$$\text{Tr}(\gamma_{0,5})^2 - 64 \text{Tr}(\gamma_{\Omega_{\frac{N}{2}},5}^{-1} \gamma_{\Omega_{\frac{N}{2}},5}^T) + 32^2, \quad (4.26)$$

which arise from the diagrams:

In the non-compact limit we are considering here, this last contribution does not survive, as it is proportional to v_6/v_4 . The fact that it vanishes is consistent with the fact that if space is not compact, there is no restriction from charge conservation on the number of D5-branes which may be present: The analogue of Gauss' Law for the 6-form potential's field strength does not apply, as the flux lines can stretch to infinity. In the compact case, they must begin and end all within the compact volume. So this equation will be relevant only when we return to the study of global 6-form charge cancelation in the compact $K3$ examples.

Notice also in this case that the last two diagrams obviously vanish in the case when N is odd. An immediate consequence of this is that \mathbf{Z}_3 fixed points have no untwisted 6-form charge. Their presence alone will not require dynamical fivebranes.

The twisted sector tadpoles are (proportional to $(1 - 1)v_6 \int_0^\infty dl$):

$$\begin{aligned} & \sum_{\substack{k=1 \\ k \neq \frac{N}{2}}}^{N-1} \left[\frac{1}{4 \sin^2 \frac{\pi k}{N}} \text{Tr}(\gamma_{k,9})^2 - 2 \text{Tr}(\gamma_{k,9}) \text{Tr}(\gamma_{k,5}) + 4 \sin^2 \frac{\pi k}{N} \text{Tr}(\gamma_{k,5})^2 \right] \\ & - 16 \sum_{\substack{k=1 \\ k \neq \frac{N}{2}}}^{N-1} \left[4 \cos^2 \frac{\pi k}{N} \text{Tr}(\gamma_{\Omega_k,5}^{-1} \gamma_{\Omega_k,5}^T) + \frac{1}{4 \sin^2 \frac{\pi k}{N}} \text{Tr}(\gamma_{\Omega_k,9}^{-1} \gamma_{\Omega_k,9}^T) \right] \\ & + 64 \sum_{\substack{k=1 \\ k \neq \frac{N}{2}}}^{N-1} \left[\frac{\cos^2 \frac{\pi k}{N}}{\sin^2 \frac{\pi k}{N}} - \delta_{N \bmod 2,0} \right]. \end{aligned} \quad (4.27)$$

These tadpoles correspond to the following diagrams:

$$\begin{aligned} & \sum_k \left[\begin{array}{c} 9 \quad \text{---} \quad k \quad \text{---} \quad 9 \\ \text{---} \end{array} \right] + \begin{array}{c} 9 \quad \text{---} \quad k \quad \text{---} \quad 5 \\ \text{---} \end{array} + \begin{array}{c} 5 \quad \text{---} \quad k \quad \text{---} \quad 5 \\ \text{---} \end{array} \\ & + \sum_k \left[\begin{array}{c} \Omega_k \quad \text{---} \quad 2k \quad \text{---} \quad 5 \\ \text{---} \end{array} + \begin{array}{c} \Omega_k \quad \text{---} \quad 2k \quad \text{---} \quad 9 \\ \text{---} \end{array} + \right. \\ & \left. \begin{array}{c} \Omega_k \quad \text{---} \quad 2k \quad \text{---} \quad \Omega_k \\ \text{---} \end{array} + \begin{array}{c} \Omega_k \quad \text{---} \quad 2k \quad \text{---} \quad \Omega_{k+\frac{N}{2}} \\ \text{---} \end{array} \right] \end{aligned}$$

Notice that since α_N^k and $\alpha_N^{k+N/2}$ both square to the same element, α_N^{2k} , we can make opposite phase choices in the composition algebra of the γ_{Ω_k} matrices:

$$\begin{aligned}\mathrm{Tr}[\gamma_{\Omega_k,9}^{-1}\gamma_{\Omega_k,9}^T] &= \mathrm{Tr}[\gamma_{2k,9}] \\ \mathrm{Tr}[\gamma_{\Omega_{k+\frac{N}{2}},9}^{-1}\gamma_{\Omega_{k+\frac{N}{2}},9}^T] &= -\mathrm{Tr}[\gamma_{2k,9}]\end{aligned}\tag{4.28}$$

for D9-branes and

$$\begin{aligned}\mathrm{Tr}[\gamma_{\Omega_k,5}^{-1}\gamma_{\Omega_k,5}^T] &= -\mathrm{Tr}[\gamma_{2k,5}] \\ \mathrm{Tr}[\gamma_{\Omega_{k+\frac{N}{2}},5}^{-1}\gamma_{\Omega_{k+\frac{N}{2}},5}^T] &= \mathrm{Tr}[\gamma_{2k,5}]\end{aligned}\tag{4.29}$$

for D5-branes. This is more than an aesthetic choice, as the first line of each of these conditions is simply the crucial result derived in ref.[16] that $\Omega^2=1$ in the 99 sector, but -1 in the 55 sector. The second line in each is the statement that $\gamma_{N/2}^2 = -1$ in each sector.

With (4.28) and (4.29), the expression (4.27) can be factorised, for even N :

$$\begin{aligned} & \sum_{k=1}^{\frac{N}{2}} \frac{1}{4 \sin^2 \frac{(2k-1)\pi}{N}} \left[\text{Tr}(\gamma_{2k-1,9}) - 4 \sin^2 \frac{(2k-1)\pi}{N} \text{Tr}(\gamma_{2k-1,5}) \right]^2 \\ & \sum_{k=1}^{\frac{N}{2}} \frac{1}{4 \sin^2 \left(\frac{2\pi k}{N} \right)} \left[\text{Tr}(\gamma_{2k,9}) - 4 \sin^2 \left(\frac{2\pi k}{N} \right) \text{Tr}(\gamma_{2k,5}) - 32 \cos \left(\frac{2\pi k}{N} \right) \right]^2 \end{aligned} \quad (4.30)$$

and for odd N :

$$\sum_{k=1}^{M-1} \frac{1}{4 \sin^2 \left(\frac{2\pi k}{N} \right)} \left[\text{Tr}(\gamma_{2k,9}) - 4 \sin^2 \left(\frac{2\pi k}{N} \right) \text{Tr}(\gamma_{2k,5}) - 32 \cos^2 \frac{\pi k}{N} \right]^2 \quad (4.31)$$

Having extracted the divergences and factorised them, revealing the tadpole equations (which may be also interpreted as charge cancelation equations, as discussed earlier) we are ready to find ways of solving these equations for the various orientifold groups.

5. $K3$ Orientifolds

5.1. The Orientifold Models and T -Duality.

Compact manifolds which can be constructed as T^4/\mathbf{Z}_N (as described in section 2) exist only for $N = 2, 3, 4$ and 6. From the discussion in section 3, we can therefore construct orientifolds of type A for all these N , but of type B only for $N = 2, 4$ and 6. We list below explicitly the orientifold groups:

$$\begin{aligned}
\mathbf{Z}_2^A &= \{1, \alpha_2^1, \Omega, \Omega\alpha_2^1\}, & \mathbf{Z}_2^B &= \{1, \Omega\alpha_2^1\}, \\
\mathbf{Z}_3^A &= \{1, \alpha_3^1, \alpha_3^2, \Omega, \Omega\alpha_3^1, \Omega\alpha_3^2\}, \\
\mathbf{Z}_4^A &= \{1, \alpha_4^1, \alpha_4^2, \alpha_4^3, \Omega, \Omega\alpha_4^1, \Omega\alpha_4^2, \Omega\alpha_4^3\}, & \mathbf{Z}_4^B &= \{1, \alpha_4^2, \Omega\alpha_4^1, \Omega\alpha_4^3\}, \\
\mathbf{Z}_6^A &= \{1, \alpha_6^1, \dots, \alpha_6^5, \Omega, \Omega\alpha_6^1, \dots, \Omega\alpha_6^5\}, \\
\mathbf{Z}_6^B &= \{1, \alpha_6^2, \alpha_6^4, \Omega\alpha_6^1, \Omega\alpha_6^3, \Omega\alpha_6^5\},
\end{aligned} \tag{5.1}$$

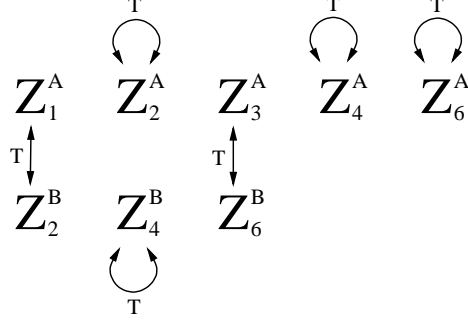
where $\alpha_N^{\frac{N}{2}} \equiv R$. In equation (4.25) for the untwisted 10-form potential, $\text{Tr}(\gamma_{0,9}) = n_9$, the number of D9-branes. All of the orientifold groups of type A contain the element Ω , and therefore there will be an equation of the form (4.25), telling us that there are 32 D9-branes. Similarly, all type A models except \mathbf{Z}_3^A will contain 32 D5-branes also, as the presence of an element ΩR means that there will be an equation of the form (4.26).

In contrast, the models of type B all lack the element Ω and therefore have only the first term of equation (4.25). Therefore the number of D9-branes in these models is zero. All type B models except \mathbf{Z}_4^B have the element ΩR , and therefore have 32 D5-branes. So \mathbf{Z}_4^B has the distinction of having no open string sectors at all: It is a consistent unoriented closed string theory⁴.

As already discussed in section 3, T -duality in the $(6, 7, 8, 9)$ directions exchanges the elements Ω and ΩR . This also exchanges D9-branes with D5-branes. So one might imagine that there are some T -duality relations amongst the models, which is of course true: Models \mathbf{Z}_2^A , \mathbf{Z}_4^A , \mathbf{Z}_6^A and \mathbf{Z}_4^B are self T_{6789} -dual. They contain the same numbers of D-branes of each type. Meanwhile \mathbf{Z}_3^A , which has only D9-branes, is dual to \mathbf{Z}_6^B which has only D5-branes. \mathbf{Z}_2^B , which has only ΩR as a non-trivial element of its orientifold group, is dual to ordinary Type I string theory (which we may denote as \mathbf{Z}_1^A), whose orientifold group has only Ω as its non-trivial element.

To summarise, we have the following picture for the T -duality relationships among the models:

⁴ There are other examples in the literature[28], constructed as an orientifold together with a translation. Later in the paper, we will find a surprise: When we compute the spectrum of our closed string model, it is the same as that of the closed string sector of the orientifold model of ref.[20].



Having established the models we wish to consider, let us revisit the tadpole equations and study them some more.

At this stage, the notation with which we compactly carried out the tadpole calculation to this point is a now more of a hindrance than an aid to clarity. Much is to be gained by simply writing out the tadpole equations explicitly in each case.

For purposes of comparison, we start with the already computed[16] \mathbf{Z}_2^A case, for which there is one twisted tadpole equation (recall $\alpha_2^1 \equiv R$):

$$\text{Tr}[\gamma_{1,9}] - 4\text{Tr}[\gamma_{1,5}] = 0. \quad (5.2)$$

The basic solution was found to be

$$\begin{aligned} \gamma_{\Omega,9} &= \gamma_{\Omega R,5} = I_{32} \\ \gamma_{R,9} &= \gamma_{\Omega,5} = \begin{pmatrix} 0 & I_{16} \\ -I_{16} & 0 \end{pmatrix}. \end{aligned} \quad (5.3)$$

For the \mathbf{Z}_3^A case we have

$$\begin{aligned} \text{Tr}[\gamma_{1,9}] - 3\text{Tr}[\gamma_{1,5}] &= 8 \\ \text{Tr}[\gamma_{2,9}] - 3\text{Tr}[\gamma_{2,5}] &= 8, \end{aligned} \quad (5.4)$$

and since we have already learned that the number of D5-branes is zero in this case, we have solution $\gamma_5 = 0$ for all orientifold elements in the D5-brane sector, and we can write

$$\begin{aligned} \gamma_{\Omega,9} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes I_8, \\ \gamma_{1,9} &= \text{diag}\{e^{\frac{2\pi i}{3}} \text{ (8 times)}, e^{-\frac{2\pi i}{3}} \text{ (8 times)}, 1 \text{ (16 times)}\}, \end{aligned} \quad (5.5)$$

from which it is trivially verified that (5.4) is satisfied.

Notice that γ_{Ω} acts by exchanging the roots and their complex conjugates: $e^{\frac{2\pi i}{3}} \leftrightarrow e^{-\frac{2\pi i}{3}}$. This will be the case in all of the later models, and so we will no longer list it explicitly

in the later solutions. Note also that in the other type A orientifolds, $\gamma_{\Omega,9} = \gamma_{\Omega R,5}$, and $\gamma_{\Omega R,9} = \gamma_{\Omega,5}$. It can also be shown that we can choose a phase such that we can always write $\gamma_{1,9} = e^{\frac{2\pi i m}{N}} \gamma_{1,5}$, for m any odd integer. That we can find such a simple relationship between the γ matrices in the D5- and D9-brane sectors is a manifestation of T_{6789} duality.

For the \mathbf{Z}_4^A case we have:

$$\begin{aligned}\text{Tr}[\gamma_{1,9}] - 2\text{Tr}[\gamma_{1,5}] &= 0 \\ \text{Tr}[\gamma_{2,9}] - 4\text{Tr}[\gamma_{2,5}] &= 0 \\ \text{Tr}[\gamma_{3,9}] - 2\text{Tr}[\gamma_{3,5}] &= 0.\end{aligned}\tag{5.6}$$

Note that the middle case correctly reproduces the \mathbf{Z}_2^A example, and therefore the \mathbf{Z}_2^A example appears as a substructure. This will be true for \mathbf{Z}_6^A also.

The solution is ($\alpha_4^2 \equiv R$)

$$\gamma_{1,9} = \text{diag}\{e^{\frac{\pi i}{4}} \text{ (8 times)}, e^{-\frac{\pi i}{4}} \text{ (8 times)}, e^{\frac{3\pi i}{4}} \text{ (8 times)}, e^{-\frac{3\pi i}{4}} \text{ (8 times)}\}\tag{5.7}$$

For the \mathbf{Z}_6 case we have:

$$\begin{aligned}\text{Tr}[\gamma_{1,9}] - \text{Tr}[\gamma_{1,5}] &= 0 \\ \text{Tr}[\gamma_{2,9}] - 3\text{Tr}[\gamma_{2,5}] &= 16 \\ \text{Tr}[\gamma_{3,9}] - 4\text{Tr}[\gamma_{3,5}] &= 0 \\ \text{Tr}[\gamma_{4,9}] - 3\text{Tr}[\gamma_{4,5}] &= -16 \\ \text{Tr}[\gamma_{5,9}] - \text{Tr}[\gamma_{5,5}] &= 0,\end{aligned}\tag{5.8}$$

for which we have ($\alpha_4^2 \equiv R$)

$$\begin{aligned}\text{Tr}[\gamma_{1,9}] = \text{Tr}[\gamma_{1,5}] &= 0, \text{Tr}[\gamma_{3,9}] = \text{Tr}[\gamma_{3,5}] = 0, \text{Tr}[\gamma_{5,9}] = \text{Tr}[\gamma_{5,5}] = 0, \\ \text{Tr}[\gamma_{2,9}] = \text{Tr}[\gamma_{2,5}] &= -8, \text{Tr}[\gamma_{4,9}] = \text{Tr}[\gamma_{4,5}] = 8, \\ \gamma_{1,9} = \text{diag}\{e^{\frac{\pi i}{6}} \text{ (4 times)}, e^{-\frac{\pi i}{6}} \text{ (4 times)}, \\ &e^{\frac{5\pi i}{6}} \text{ (4 times)}, e^{-\frac{5\pi i}{6}} \text{ (4 times)}, -i \text{ (8 times)}, i \text{ (8 times)}\}.\end{aligned}\tag{5.9}$$

Note here that

$$\gamma_{1,9} \equiv \text{diag}\{e^{\frac{2\pi i}{3}} \text{ (4 times)}, e^{-\frac{2\pi i}{3}} \text{ (4 times)}, 1 \text{ (8 times)}\} \otimes \text{diag}\{-i, i\},\tag{5.10}$$

which shows a $\mathbf{Z}_3 \times \mathbf{Z}_2$ structure, using the solutions previously obtained for the \mathbf{Z}_2^A and \mathbf{Z}_3^A models.

Also notice that in all cases above, the coefficient of the $\gamma_{k,5}$ trace is the square root of the number of fixed points invariant under α_N^k . Also interesting is that (generalising the \mathbf{Z}_2 case,) the same choice made for D9-branes can be made for D5-branes, up to a phase.

The tadpoles for the case \mathbf{Z}_6^B will turn out to be isomorphic to those listed above for \mathbf{Z}_3^A , while there are no tadpoles to list for the \mathbf{Z}_4^B model, as there are no D-branes required.

Let us now turn to the closed string spectra.

5.2. Closed String Spectra

The right moving untwisted sector has the massless states:

Sector	State	α_N^k	$SO(4)$ rep.
NS :	$\psi_{-1/2}^\mu 0\rangle$	1	(2, 2)
	$\psi_{-1/2}^{1\pm} 0\rangle$	$e^{\pm \frac{2\pi i k}{N}}$	2 (1, 1)
	$\psi_{-1/2}^{2\pm} 0\rangle$	$e^{\mp \frac{2\pi i k}{N}}$	2 (1, 1)
R :	$ s_1 s_2 s_3 s_4\rangle$		
	$s_1 = +s_2, s_3 = +s_4$	1	2 (2, 1)
	$s_1 = -s_2, s_3 = -s_4$	$e^{\pm \frac{2\pi i k}{N}}$	2 (1, 2)

(5.11)

while the right moving sector twisted by $\frac{m}{N} \neq \frac{1}{2}$ has:

Sector	State	α_N^k	$SO(4)$ rep.
NS :	$\psi_{-1/2+m/N}^{1+} 0\rangle$	$e^{\frac{2\pi i k}{N}(1-2\frac{m}{N})}$	(1, 1)
	$\psi_{-1/2+m/N}^{2+} 0\rangle$	$e^{\frac{2\pi i k}{N}(1-2\frac{m}{N})}$	(1, 1)
R :	$ s_1 s_2\rangle, s_1 = -s_2$	$e^{\frac{2\pi i k}{N}(1-2\frac{m}{N})}$	(1, 2) .

(5.12)

The exception to this situation is when we have a $\frac{m}{N} = \frac{1}{2}$ twist:

Sector	State	α_N^k	$SO(4)$ rep.
NS :	$ s_3 s_4\rangle, s_3 = +s_4$	1	2 (1, 1)
R :	$ s_1 s_2\rangle, s_1 = -s_2$	1	(1, 2) .

(5.13)

We have imposed the GSO projection, and decomposed the little group of the spacetime Lorentz group as $SO(4) = SU(2) \times SU(2)$. We form the spectrum for orientifold group of type A by taking products of states from the left and right sectors (to give states invariant under α_N^k), symmetrised by the Ω projection in the NS-NS sector, while antisymmetrising in the R-R sector.

Finally, we have from the untwisted closed string sector of the type A \mathbf{Z}_N orientifold ($N \neq 2$):

Sector	$SO(4)$ rep.
NS NS :	(3, 3) + 5(1, 1)
R R :	(3, 1) + (1, 3) + 4(1, 1) .

(5.14)

This is the content of the $\mathcal{N}=1$ supergravity multiplet in six dimensions, accompanied by one tensor multiplet and 2 hypermultiplets. In the case of \mathbf{Z}_2 it is [16]:

Sector	$SO(4)$ rep.
NS NS :	(3, 3) + 11(1, 1)
R R :	(3, 1) + (1, 3) + 6(1, 1) ,

(5.15)

the $D=6$, $\mathcal{N}=1$ supergravity multiplet in six dimensions, accompanied by one tensor multiplet and 4 hypermultiplets

The twisted sectors will produce additional multiplets. The bosonic content of a hypermultiplet is four scalars $4(\mathbf{1}, \mathbf{1})$, while that of a tensor multiplet is $(\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1})$. By combining on the left and right the sectors twisted by $\frac{m}{N}$ and $(1 - \frac{m}{N})$, we find that the NS-NS sector produces one hypermultiplet while the R-R sector produces a tensor multiplet. A sector twisted by $\frac{1}{2}$ simply produces one hypermultiplet: one quarter coming from the R-R sector and three quarters from the NS-NS sector.

To evaluate[25] the number of hypermultiplets coming from the twisted sectors of a $K3$ orbifold, we need to recall the structure of the fixed points and their transformation properties, as reviewed in section 2. In the case of \mathbf{Z}_2^A , we simply multiply by the number of \mathbf{Z}_2 fixed points, and we find that there are 16 hypermultiplets from the twisted sectors, giving a total of 20 hypermultiplets when combined with the four from the untwisted sector.

For \mathbf{Z}_3^A there are 9 fixed points, each supplying a hypermultiplet and a tensor multiplet, (for twists by $(\frac{1}{3}, \frac{2}{3})$), giving a total of 11 hypermultiplets and 10 tensor multiplets when added to those arising in the untwisted sector.

For \mathbf{Z}_4^A the four \mathbf{Z}_4 invariant fixed points give 4 hypermultiplets and four tensor multiplets. They are also \mathbf{Z}_2 fixed points and so supply an additional 4 hypermultiplets. The other 12 \mathbf{Z}_2 fixed points form 6 \mathbf{Z}_4 invariant pairs, from which arise 6 hypermultiplets. This gives a total of 16 hypermultiplets and 5 tensor multiplets for the complete model.

Finally, for the model \mathbf{Z}_6^A , the \mathbf{Z}_6 fixed point gives 2 hypermultiplet and 2 tensor multiplet from $(\frac{1}{6}, \frac{5}{6})$ and $(\frac{1}{3}, \frac{2}{3})$ twists. It also gives 6 hypermultiplets from the $\frac{1}{2}$ twisted sector. The 4 pairs of \mathbf{Z}_3 points give 4 tensor multiplets and 4 hypermultiplets for $(\frac{1}{3}, \frac{2}{3})$ twists, while the 5 \mathbf{Z}_2 triplets of fixed points supply 5 hypermultiplets. This gives 14 hypermultiplets and 7 tensor multiplets in all.

For \mathbf{Z}_N orientifolds of type B the situation is as follows. For closed string states, prior to making the theory unorientable, the relevant orbifold states to consider are those for the group formed by the remaining pure \mathbf{Z}_N elements in the orientifold group, which is therefore $\mathbf{Z}_{\frac{N}{2}}$. The possible left and right states are evaluated as before, and then they are projected to the unoriented theory invariant under $\Omega \cdot \alpha_N^1$.

It is thus easy to see that the closed string spectra for \mathbf{Z}_6^B and \mathbf{Z}_3^A are isomorphic, as are those of \mathbf{Z}_2^B and \mathbf{Z}_1^A , (the latter being simply ten dimensional Type I string theory: there is no orbifold to perform for \mathbf{Z}_2^B).

There remains only the spectrum of \mathbf{Z}_4^B to compute, which is self T-dual. The pure orbifold states to consider are those of \mathbf{Z}_2 . Tensoring left and right to form the Ω_1 invariant spectrum, we obtain 12 hypermultiplets and 9 tensor multiplets in total.

In summary, we have (in addition to the usual gravity and tensor multiplet) the following

spectrum of hypermultiplets and tensor multiplets from the closed string sector for each model:

Model	Neutral Hypermultiplets	Extra Tensor Multiplets
\mathbf{Z}_2^A	20	0
\mathbf{Z}_3^A	11	9
\mathbf{Z}_4^A	16	4
\mathbf{Z}_6^A	14	6
\mathbf{Z}_4^B	12	8
\mathbf{Z}_6^B	11	9

Of course, the fact that we have obtained, in addition to the usual supergravity multiplet and tensor multiplet, a total of 20 hypermultiplets plus tensor multiplets (80 scalar fields) in the different orbifold limits of $K3$ should alert us that we are computing an invariant property of the manifold: there are 80 moduli of the $K3$ surface[29]. It is a four dimensional manifold and so they should naturally combine into 20 hypermultiplets if all of these moduli were available to us. This is what happened in the case of \mathbf{Z}_2^A , as can be seen above. However, in the other other orientifold examples, some of the moduli scalars combine into tensor multiplets, leaving us with fewer hypermultiplets in the final model, presumably corresponding to a reduction in the dimension of the moduli space of $K3$ deformations available to these models.

We now turn to the open string spectra.

5.3. Open String Spectra

Let us study first the 99 open string sector. The massless bosonic spectrum arises as follows:

$$\begin{array}{ccccccc}
\text{state} & \alpha_N^k = + & \Omega = + & SO(4) \text{ rep.} & & & \\
99 \left\{ \begin{array}{lll} \psi_{-1/2}^\mu |0, ij\rangle \lambda_{ij} & \lambda = \gamma_{k,9} \lambda \gamma_{k,9}^{-1} & \lambda = -\gamma_{\Omega,9} \lambda^T \gamma_{\Omega,9}^{-1} & (\mathbf{2}, \mathbf{2}) \\ \psi_{-1/2}^{1\pm} |0, ij\rangle \lambda_{ij} & \lambda = e^{\pm \frac{2\pi k}{N}} \gamma_{k,9} \lambda \gamma_{k,9}^{-1} & \lambda = -\gamma_{\Omega,9} \lambda^T \gamma_{\Omega,9}^{-1} & 2(\mathbf{1}, \mathbf{1}) \\ \psi_{-1/2}^{2\pm} |0, ij\rangle \lambda_{ij} & \lambda = e^{\mp \frac{2\pi k}{N}} \gamma_{k,9} \lambda \gamma_{k,9}^{-1} & \lambda = -\gamma_{\Omega,9} \lambda^T \gamma_{\Omega,9}^{-1} & 2(\mathbf{1}, \mathbf{1}) \end{array} \right. & & & (5.16)
\end{array}$$

For the 55 states at a fixed point we have:

$$\begin{array}{ccccc}
\text{state} & \alpha_N^k = + & \Omega = + & SO(4) & \text{rep.} \\
55 \left\{ \begin{array}{lll} \psi_{-1/2}^\mu |0, ij\rangle \lambda_{ij} & \lambda = \gamma_{k,5} \lambda \gamma_{k,5}^{-1} & \lambda = -\gamma_{\Omega,5} \lambda^T \gamma_{\Omega,5}^{-1} & \mathbf{(2, 2)} \\ \psi_{-1/2}^{1\pm} |0, ij\rangle \lambda_{ij} & \lambda = e^{\pm \frac{2\pi k}{N}} \gamma_{k,5} \lambda \gamma_{k,5}^{-1} & \lambda = \gamma_{\Omega,5} \lambda^T \gamma_{\Omega,5}^{-1} & 2(\mathbf{1, 1}) \\ \psi_{-1/2}^{2\pm} |0, ij\rangle \lambda_{ij} & \lambda = e^{\mp \frac{2\pi k}{N}} \gamma_{k,5} \lambda \gamma_{k,5}^{-1} & \lambda = \gamma_{\Omega,5} \lambda^T \gamma_{\Omega,5}^{-1} & 2(\mathbf{1, 1}) \end{array} \right. & & & & (5.17)
\end{array}$$

For the 55 states away from a fixed point we have:

$$\begin{array}{ccccc}
\text{state} & \Omega = + & SO(4) & \text{rep.} \\
55 \left\{ \begin{array}{lll} \psi_{-1/2}^\mu |0, ij\rangle \lambda_{ij} & \lambda = -\gamma_{\Omega,5} \lambda^T \gamma_{\Omega,5}^{-1} & \mathbf{(2, 2)} \\ \psi_{-1/2}^{1\pm} |0, ij\rangle \lambda_{ij} & \lambda = \gamma_{\Omega,5} \lambda^T \gamma_{\Omega,5}^{-1} & 2(\mathbf{1, 1}) \\ \psi_{-1/2}^{2\pm} |0, ij\rangle \lambda_{ij} & \lambda = \gamma_{\Omega,5} \lambda^T \gamma_{\Omega,5}^{-1} & 2(\mathbf{1, 1}) \end{array} \right. & & & & (5.18)
\end{array}$$

For the 59 states we have at a fixed point:

$$\begin{array}{ccccc}
\text{state} & \alpha_N^k = + & SO(4) & \text{rep.} \\
59 : & |s_3 s_4, ij\rangle \lambda_{ij}, s_3 = s_4 & \lambda = \gamma_{k,5} \lambda \gamma_{k,9}^{-1} & \mathbf{2(1, 1)} & (5.19)
\end{array}$$

and away from a fixed point:

$$\begin{array}{ccccc}
\text{state} & SO(4) & \text{rep.} \\
59 : & |s_3 s_4, ij\rangle \lambda_{ij}, s_3 = s_4 & 2(\mathbf{1, 1}) & & (5.20)
\end{array}$$

Using the solution presented in section 4 for the γ matrices, we find the following solutions for the open string spectra of the models:

Model	Gauge Group	Charged Hypermultiplets
\mathbf{Z}_2^A	99 : $U(16)$ 55 : $U(16)$	99 : $2 \times \mathbf{120}$ 55 : $2 \times \mathbf{120}$ 59 : $(\mathbf{16}, \mathbf{16})$
\mathbf{Z}_3^A	99 : $U(8) \times SO(16)$	99 : $(\mathbf{28}, \mathbf{1}); (\mathbf{8}, \mathbf{16})$
\mathbf{Z}_4^A	99 : $U(8) \times U(8)$ 55 : $U(8) \times U(8)$	99 : $(\mathbf{28}, \mathbf{1}); (\mathbf{1}, \mathbf{28}); (\mathbf{8}, \mathbf{8})$ 55 : $(\mathbf{28}, \mathbf{1}); (\mathbf{1}, \mathbf{28}); (\mathbf{8}, \mathbf{8})$ 59 : $(\mathbf{8}, \mathbf{1}; \mathbf{8}, \mathbf{1}); (\mathbf{1}, \mathbf{8}; \mathbf{1}, \mathbf{8})$
\mathbf{Z}_6^A	99 : $U(4) \times U(4) \times U(8)$ 55 : $U(4) \times U(4) \times U(8)$	99 : $(\mathbf{6}, \mathbf{1}, \mathbf{1}); (\mathbf{1}, \mathbf{6}, \mathbf{1})$ $(\mathbf{4}, \mathbf{1}, \mathbf{8}); (\mathbf{1}, \mathbf{4}, \mathbf{8})$ 55 : $(\mathbf{6}, \mathbf{1}, \mathbf{1}); (\mathbf{1}, \mathbf{6}, \mathbf{1})$ $(\mathbf{4}, \mathbf{1}, \mathbf{8}); (\mathbf{1}, \mathbf{4}, \mathbf{8})$ 59 : $(\mathbf{4}, \mathbf{1}, \mathbf{1}; \mathbf{4}, \mathbf{1}, \mathbf{1})$ $(\mathbf{1}, \mathbf{4}, \mathbf{1}; \mathbf{1}, \mathbf{4}, \mathbf{1})$ $(\mathbf{1}, \mathbf{1}, \mathbf{8}; \mathbf{1}, \mathbf{1}, \mathbf{8})$
\mathbf{Z}_4^B	—	—
\mathbf{Z}_6^B	55 : $U(8) \times SO(16)$	55 : $(\mathbf{28}, \mathbf{1}); (\mathbf{8}, \mathbf{16})$

6. Small Instantons and the Dynamics of Fivebranes

In the table above, we have given the spectrum for configurations with all of the D5–branes sitting on a single fixed point. There are other models corresponding to different configurations. It is easy to see that some configurations can be reached from other models in the same moduli space: For fixed the A type orientifolds, a group of $2N$ D5–branes can move off a fixed point together, forming a ‘dynamical fivebrane’. The 55 hypermultiplets supply precisely the structure needed to act as moduli for the process of moving off a fixed point.

For example, in the case of \mathbf{Z}_4 , four dynamical five–branes (8 D5–branes each) can to move off the fixed point into the bulk, breaking the group to a $U(6) \times U(6)$ factor from the remaining 24 D5–branes on the fixed point and a $SU(2)$ factor from the fivebrane in bulk. If $m \leq 4$ of these fivebranes move off, the group factor at the fixed point is $U(8 - 2m) \times U(8 - 2m)$, and away from it is $SU(2)^m$ for m separated fivebranes, or

$USp(2m)$ if they coincide. It can also be $U(m) \times U(m)$ if the fivebranes move to another fixed point. The spectra for these configurations are accompanied by hypermultiplets in the obvious analogous representations from the 55 and 59 sectors. The resulting gauge groups and hypermultiplets are precisely those obtained from the Higgs mechanism, whereby some of the gauge fields swallow hypermultiplets from the 55 sector.

For the case Z_6 the dynamics of fivebranes is as follows. The hypermultiplet structure permits the Higgs mechanism corresponding to the movement of 2 dynamical hypermultiplets (12 D5-branes each). A single such object at a fixed point gives $U(2) \times U(2) \times U(2)$ contribution to the gauge group. Away from a fixed point it is again $SU(2)$.

Notice that in this case there is always a residue of 8 D5-branes (equivalent to $2/3$ five-brane) which must sit at either all at the Z_6 fixed point, or distributed amongst the Z_3 points in a Z_6 and Ω invariant way⁵. The first case gives a factor $U(4)$, while the two ways of doing the latter give $U(2) \times U(2)$ and $U(1)^4$ (The minimum number of D5-branes at any given point is 2, due to the Ω projection.)

The enhanced gauge symmetries associated to the isolated or coincident dynamical fivebranes are precisely the structures observed in ref.[12]. The heterotic fivebrane/small instantons in the dual heterotic model (expected to exist for each of the models presented here) is realised in the Z_N^A orientifold as a family of $2N$ D5-branes forced to move together as one when away from fixed points.

There are disconnected families of models formed by distributing the D5-branes among different fixed points in a Z_N invariant way. Once again (if possible) complete fivebranes can move off the fixed points, giving $SU(2)$ and $USp(m)$ factors from the bulk as usual, while leaving behind any group of D5-branes which is less than $2N$. This remainder will produce unitary factors to the gauge group in the obvious way, with $U(1)$ arising for each isolated pair on a fixed point, etc.

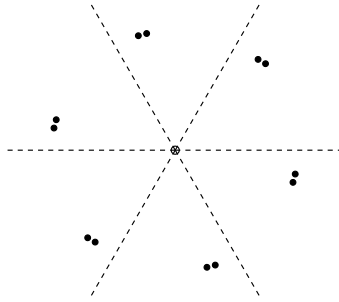
Let us consider the dynamics of fivebranes a little more, in the light of the fixed point structure of the orbifolds. D5-branes may move off fixed points in groups of $2N$. This is accomplished by recalling the transformation properties of the fixed points under the Z_N generator, discussed in section 2. Taking the most complicated example, $K3(Z_6)$, we see that twelve D5-branes may move off the Z_6 fixed point together. Also allowed is for six D5-branes to move off a Z_3 fixed point at the same time as another six move off its doublet partner. Similarly, a triplet of Z_2 fixed points can each eject four D5-branes simultaneously. These are all of the ways in which a dynamical fivebrane may venture into open space, away from the ALE singularities of the $K3(Z_6)$ orbifold.

⁵ This is related to the structure of the Z_6 equations in section 4. There was a non-zero trace for the γ matrices, corresponding to some number of branes which could not be grouped in such a way as to define an object uncharged under the twisted R-R six-forms. Notice also that there are 8 identical eigenvalues of the $\gamma_{1,5}$ matrix.

The $K3(\mathbf{Z}_4)$ example is more straightforward. A \mathbf{Z}_4 ALE point can yield eight D5–branes, the basic dynamical unit in this example. Alternatively, one of the six pairs of \mathbf{Z}_2 fixed points can produce the same type of object.

The 55 sector of the $K3(\mathbf{Z}_3)$, as discussed before, is trivial: There are no D5–branes.

D5–branes in the bulk (away from fixed points) are only subject to the Ω projection. The $2N$ components of the dynamical fivebrane, have split into N pairs, one in each sector of the local division of \mathbf{R}^4 into slices performed by the rotation generated by \mathbf{Z}_N . The orbifold ensures that their movements are correlated, however, as α_N relates each sector. For example for $K3(\mathbf{Z}_6)$, we have the following picture:



where the central point is either a \mathbf{Z}_6 ALE singularity, or a doublet of \mathbf{Z}_3 ALE singularities, or a triplet of \mathbf{Z}_2 ALE points. In the latter two cases, those points have been identified in order to form the \mathbf{Z}_6 –invariant fundamental domain. The pair of points⁶ is a D5–brane pair, forced to move together by the orientifold[16].

For the \mathbf{Z}_6^B model, the only non–trivial B –type model with D5–branes, we have a slightly different situation. The spectrum is the same (by T–duality) as the model \mathbf{Z}_3^A where the gauge group is carried by D9–branes. Here, however, the gauge group is carried by D5–branes on a single fixed point. We can connect to other models by Higgsing, moving dynamical fivebranes off the fixed point as before. Also as before, we can make families in disconnected sectors of moduli space by making different starting configurations of distributions of D5–branes.

However, there are some important differences. First, the number of D5–branes making up one fivebrane is 6. This can be seen easily from the fact that (as mentioned before) the spacetime orbifold group is not \mathbf{Z}_N , but $\mathbf{Z}_{N/2}$ for the \mathbf{Z}_N^B orientifold group. Therefore the D5–branes are forced to move as triplets by the \mathbf{Z}_3 and these triplets are forced to move in correlated pairs by the orientifold.

⁶ Of course, we have only drawn one of the two complex planes of the original T^4 here. Furthermore, the worldvolume of the D5–branes under discussion here fill out the non–compact six dimensional spacetime.

Another difference between \mathbf{Z}_6^B and the A -type models is that the gauge group of a single fivebrane (6 D5-branes) is $U(1)$ and not $SU(2)$. The reason can be seen as follows: In the A -type models the orientifold group element Ω correlated the movements of a pair of D5-branes, and in addition constrained them to be at the same position on the torus[16]. Here however, the orientifold group element ΩR appears instead of Ω and it correlates the dynamics of a pair of branes, as before, but they are not coincident, being mirror images of each other under of the action of R . A quick check of the Higgs mechanism using the spectrum given in the table shows that indeed $U(1)$ is the correct gauge group for an isolated dynamical fivebrane in this model, and not $SU(2)$. This is also consistent with the fact that completely isolated D-branes should carry $U(1)$ gauge group, and no more.

There are 4 possible dynamical fivebranes in this model. Starting with all of the D5-branes on one point, with gauge group $U(8) \times SO(16)$ we can move them off in m groups of 6 to give a breaking to $U(8 - 2m) \times SO(16 - 2m)$, for $m < 5$, with factor $U(1)^m$ if the fivebranes are all isolated, and $U(m)$ if all coincident. (The intermediate cases are obvious.) Once $SO(8) \times U(1)^4$ has been reached corresponding to 8 D5-branes remaining on the fixed point, and 24 in bulk (forming 4 fivebranes) no more groups of 6 can be moved off⁷.

Let us briefly consider the blow-up limits of the \mathbf{Z}_N^A models which we have been considering. As we reminded ourselves in section 2, the smooth $K3$ limit is obtained by performing surgery on the orbifold limits we have discussed here, where the singular ALE points are excised and replaced by small ALE gravitational instantons⁸. It is interesting to consider how some of the results obtained in the rest of the paper fit in with this process.

Considering first the \mathbf{Z}_2^A of ref.[16], we have 32 D5-branes, which are grouped into units of 4 to make 8 dynamical fivebranes, corresponding to one small instanton on the heterotic side[12]. The total instanton number (contribution to $\int F \wedge F$) of $K3$ is 24, and so the 16 small \mathcal{E}_2 gravitational instantons from the blow-up must each supply one unit. Meanwhile, the Euler number (contribution to $\int R \wedge R$) of $K3$ is 24. As the ALE spaces supply all of the curvature for the smooth manifold, they must each contribute $3/2$ to the total. The (untwisted) R-R field strength H_7 of the R-R six-form A_6 (which naturally couples to the world volume of a fivebrane) must obey total cancelation of its charge $Q_5 \propto (\int F \wedge F - \int R \wedge R)$ in the compact internal $K3$ space, which it does with the given assignments above. This gives the H_7 charge of a single \mathcal{E}_2 space as $\tilde{\mu}_5^{(2)} = 1 - 3/2 = -1/2$, measured in units where we set the dynamical fivebrane's H_7 -charge to one⁹. The solution of the tadpole

⁷ This can be seen in many different ways. One way involves noting that (as happened in the \mathbf{Z}_6^A case, there are eight identical eigenvalues of the $\gamma_{2,5}$ matrix, corresponding to the number of branes which cannot be moved off.

⁸ We have not considered in any detail the whereabouts of the moduli for this process. They should exist, and we assume that they do for this brief discussion.

⁹ This is not just a natural or convenient choice, but the physical one: We must not forget Dirac–Teitelboim–Nepomechie quantisation of the six-form charge. In type IIB, the basic units of

equation (4.26) by choosing 32 D5-branes is simply reinterpreted here as an equation for conservation of H_7 -charge: $16\tilde{\mu}_5^{(2)} + 8\mu_5 = 0$.

Let us see how this works in the other models. In the \mathbf{Z}_3^A case, we learn something simple. Recall that there are no D5-branes in this case. This means that H_7 charge must be zero due to exact cancelation of the ALE points' instanton number against Euler number. As there were 9 identical fixed points, we see that the contribution to Euler number of an \mathcal{E}_3 space must be $8/3$, and their instanton number in these units must be equal and opposite to this. The H_7 charge $\tilde{\mu}_5^{(3)}$ of a \mathcal{E}_3 space is the difference between these two — zero — which is of course why there are no D5-branes; another interpretation of the absence of D5-branes, as taught to us by equation (4.26).

The \mathbf{Z}_4 case is the first of the two examples with mixed species of ALE instanton in the blow-up. It has 16 singularities to be blown up. The blow-up replaces 4 \mathbf{Z}_4 points with \mathcal{E}_4 spaces. The other 12 singularities are paired into 6 doublets and blown up with 6 \mathcal{E}_2 's. We learned previously that the contribution to Euler number of the \mathcal{E}_2 's was $3/2$ and so we deduce the contribution of an \mathcal{E}_4 to be $15/4$ after simple arithmetic. Consider now instanton number. We have 32 D5-branes, which clump into groups of 8 to make 4 dynamical fivebranes, each of which corresponds to one small heterotic instanton. We also have that one \mathcal{E}_2 space has instanton number 1. Putting this all together, and doing some more simple arithmetic, we find that $Q_5 = 0$ is satisfied when the instanton number of an \mathcal{E}_4 is $14/4$. This gives us an H_7 charge $\tilde{\mu}_5^{(4)}$ of an \mathcal{E}_4 space equal to $-1/4$.

We turn finally to the \mathbf{Z}_6 case, with its 24 fixed points. We still have 32 D5-branes, from (4.26), and these should correspond to $32/12 = 2 + 2/3$ dynamical fivebranes/small instantons. So only two fivebranes can be untethered to fixed points in this model, as we saw earlier. Moving on, we can put together all that we have learned to deduce that the combination of one \mathcal{E}_6 space (to resolve the \mathbf{Z}_6 singularity), 4 \mathcal{E}_3 spaces (to resolve the eight \mathbf{Z}_3 singularities), 5 \mathcal{E}_2 spaces (to resolve the fifteen \mathbf{Z}_3 singularities) and the $2 + 2/3$ instanton yields for \mathcal{E}_6 an Euler number contribution of $35/6$, and an instanton number contribution of $34/6$. The H_7 charge $\tilde{\mu}_5^{(6)}$ of an \mathcal{E}_6 space is therefore equal to $-1/6$.

It is quite pleasing to observe the results of the above numerical exercise. The fact that for an \mathcal{E}_m ALE space, the H_7 charge is given by $\tilde{\mu}_5^{(m)} = -1/m$ for m even and is zero for m odd is indeed what we deduced from studying and solving the tadpole equations. This also means that we can place $1/m$ of a fivebrane on each \mathcal{E}_m space to cancel all H_7 -charge *locally*, in any of the models¹⁰. As we saw above, we learned that this was possible from direct computation of the spectrum.

quantisation were carried by D5-branes. Now it is carried by $2N$ of them. This is possible because the Ω projection renormalises the basic unit of charge by $1/2$, and the \mathbf{Z}_N projection by another $1/N$. This generalises the comments made in ref.[16].

¹⁰ This is a configuration which will be of some interest to us later.

What we have done here is simply to rephrase the results we obtained in a language quite well-suited to the discussion of dual six dimensional heterotic string theory compactifications. It will be exciting to return to more of these issues in some detail.

7. Anomalies

We have verified that the irreducible $\text{tr}R^4$ terms in the gravitational anomaly polynomials vanish for these spectra, by checking that the equation

$$n_H - n_V = 244 - 29n_T \tag{7.1}$$

is satisfied. (Here, n_H, n_V and $n_T + 1$ are respectively the numbers of hypermultiplets, vector multiplets and tensor multiplets in the six dimensional supergravity model.) We have also checked that the irreducible $\text{tr}F^4$ terms in the gauge anomaly polynomials vanish.

That these anomalies cancel is a fine example of the interplay between the $K3$ geometry, the open string sectors, and closed string sectors. In the \mathbf{Z}_2^A example of ref.[16], the number n_T is zero, and the 20 closed string hypermultiplets contribute to the cancelation of the anomaly in the usual way. One might have expected that this procedure would have persisted, the closed string sector providing the 20 hypermultiplets as usual, and the open string contribution to the vector multiplets and hypermultiplets changing in such a way as to make sure that the anomaly is canceled.

A first realisation that this expectation will go spectacularly wrong is the observation that the Z_4^B model has no open string sectors: it is a purely closed string theory. So the contribution of the open string vectors and hypermultiplets is absent, and the resulting model would be terribly sick due to gravitational anomalies. However, it is seen that the number of closed string hypermultiplets reduces to 12, in exchange for 8 extra tensor multiplets, rendering the theory anomaly free, and producing a consistent closed unoriented string theory¹¹.

The same sort of thing happens for the other models, with a varying split of the number 20 between the number of hypermultiplets and extra tensor multiplets, in just such a way as to combine with the open string spectrum to give an anomaly free theory.

For the mixed anomalies, we expect that the anomalies all factorise in a way consistent with cancelation by a generalisation[19] of the Green–Schwarz mechanism. This was studied for the \mathbf{Z}_2^A model in ref.[17], and for a related model (similar to B -type models) in ref.[20]. Similar structures will arise here, and we shall present a discussion of these and related issues in a separate publication[30].

¹¹ This model is the same as the closed string sector of ref.[20], obtained while studying a different orientifold group.

8. Conclusions

We have investigated a large family of orientifold models, corresponding to consistent open and closed string unoriented string theories propagating on the $K3$ surface, yielding six dimensional string theories with $\mathcal{N}=1$ spacetime supersymmetry.

Along the way we have observed a truly remarkable interplay between three separate (but related) elements: The local behaviour of open string sectors (D-branes) in the neighbourhood of ALE points¹², as captured by the tadpole equations; the behaviour of the unoriented closed string sectors; and the geometry of the $K3$ manifold. In every case we studied, the geometry of $K3$ seemed to know exactly how to combine the 32 D9- and D5-branes with the closed string twisted sectors in such a way as to give a consistent model.

This is further confirmation that the details of orientifold construction set out in ref.[16] are a powerful addition to (and refinement of) the tools available to study $\mathcal{N}=1$ string models, reuniting open and unoriented strings with heterotic strings in the rich family of models which should be studied in this context.

The observation of refs.[12,16] that the type I string theory's basic dynamical fivebranes have a symplectic projection on their Chan-Paton factors was seen to persist in these models, and is expected to do so as long as the D5-branes are away from special points. The mechanism by which such a dynamical fivebrane arose in each model is easily traced to the spacetime and orientifold symmetries. Further to this, in each particular model, we saw that the individual ALE instanton/fixed point structures give rise to new families of enhanced gauge symmetry groups, with an associated spectrum of charged matter.

We expect that in the case of the even $N \mathbf{Z}_N^A$ models, there will be an interpretation in terms of some dual families of $SO(32)$ heterotic string models compactified on $K3$, where the D9-brane contributions will be perturbatively realised, while the D5-brane contributions will appear as non-perturbative symmetries. This will generalise the ideas in refs.[12,16,17,32]. It will be certainly interesting to investigate these models further[30].

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¹² See ref.[31] for a complementary study of D-branes and ALE singularities.

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